

# UPPER TRIANGULAR FORMS AND SPECTRAL ORDERINGS IN A $\text{II}_1$ -FACTOR

J. NOLES

**ABSTRACT.** Dykema, Sukochev and Zanin used a Peano curve covering the support of the Brown measure of an operator  $T$  in a diffuse, finite von Neumann algebra to give an ordering to the support of the Brown measure, and create a decomposition  $T = N + Q$ , where  $N$  is normal and  $Q$  is s.o.t.-quasinilpotent. In this paper we prove that a broader class of measurable functions can be used to order the support of the Brown measure giving normal plus s.o.t.-quasinilpotent decompositions.

## 1. INTRODUCTION AND DESCRIPTION OF RESULTS

We start with a famous theorem of Schur (see for instance [7]) which will motivate this paper.

**Theorem 1.** *For every matrix  $T \in M_n(\mathbf{C})$ , there exists a unitary matrix  $U \in M_n(\mathbf{C})$  such that  $U^{-1}TU$  is an upper triangular matrix.*

The diagonal entries of  $U^{-1}TU$  are the eigenvalues of  $T$ , repeated up to multiplicity, and  $U$  can be chosen so that they appear in any order. Hence each ordering of the spectrum of  $T$  gives a decomposition  $T = N + Q$ , where  $N$  is normal and  $Q$  is nilpotent.

In [3], Dykema, Sukochev and Zanin use Haagerup-Schultz projections to prove a related theorem in  $\text{II}_1$ -factors.

**Theorem 2.** *Let  $\mathcal{M}$  be a diffuse, finite von Neumann algebra with normal, faithful, tracial state  $\tau$  and let  $T \in \mathcal{M}$ . Then there exist  $N, Q \in \mathcal{M}$  such that*

- (1)  $T = N + Q$
- (2) *the operator  $N$  is normal and the Brown measure of  $N$  equals that of  $T$*
- (3) *The operator  $Q$  is s.o.t.-quasinilpotent.*

The proof of Theorem 2 uses a Peano curve  $\rho : [0, 1] \rightarrow \overline{B_{\|T\|}}$ . The normal operator  $N$  is created by taking the trace-preserving conditional expectation onto the von Neumann algebra generated by the Haagerup-Schultz projections of the operator  $T$  associated with the sets  $\rho([0, t])$  for  $t \in [0, 1]$ . These projections, along with the normal operator  $N$ , are determined by the ordering on the support of the Brown measure of  $T$  given by  $z_1 \leq z_2$  if and only if  $\min(\rho^{-1}(z_1)) \leq \min(\rho^{-1}(z_2))$ . Theorem 2 generalizes the idea of using an ordering of the spectrum of the operator  $T$  to write it as an uppertriangular form.

In this paper we will further generalize the idea of spectral orderings from the finite dimensional case to  $\text{II}_1$ -factors. We show that normal plus s.o.t.-quasinilpotent

decompositions are generated not only by continuous orderings, but by a large class of measurable orderings.

**Theorem 3.** *Let  $\mathcal{M}$  be a  $II_1$ -factor and  $T \in \mathcal{M}$ . Let  $\nu_T$  be the Brown measure of  $T$  and for a Borel set  $B \subset \overline{B_{\|T\|}}$ , let  $P_T(B)$  be the Haagerup-Schultz projection for the operator  $T$  associated to the set  $B$ . Let  $\psi : [0, 1] \rightarrow \overline{B_{\|T\|}}$  be a Borel measurable function such that  $\psi([0, t])$  is Borel for all  $t \in [0, 1]$ ,  $\{z \in \overline{B_{\|T\|}} : \psi^{-1}(z) \text{ has a minimum}\}$  is Borel, and*

$$\nu_T(\{z \in \overline{B_{\|T\|}} : \psi^{-1}(z) \text{ has a minimum}\}) = 1.$$

*Then there exists a spectral measure  $E$  supported on  $\text{supp}(\nu_T)$  such that*

- (1)  $E(\psi([0, t])) = P_T(\psi([0, t]))$  for all  $t \in [0, 1]$ ,
- (2)  $\tau(E(B)) = \nu_T(B)$  for all Borel  $B \subset \overline{B_{\|T\|}}$ , and
- (3)  $T - \int_{\mathbf{C}} z dE$  is s.o.t.-quasinilpotent.

In particular the conclusion holds if  $\psi$  is continuous or is a Borel isomorphism. We leave open the following question: Given a function  $\varphi$  which satisfies the hypotheses of Theorem 3, does there exist a Borel isomorphism  $\psi$  such that  $\varphi$  and  $\psi$  generate the same spectral measure?

Note that part 2 of theorem 3 implies that  $\int_{\mathbf{C}} z dE$  and  $T$  have that same Brown measure.

## 2. BACKGROUND: CONDITIONAL EXPECTATION, BROWN MEASURE, HAAGERUP-SCHULTZ PROJECTIONS AND S.O.T.-QUASINILPOTENT OPERATORS

This section includes some background necessary for the proof of Theorem 3. Throughout this section  $\mathcal{M}$  is a  $II_1$ -factor with trace  $\tau$ , and  $T \in \mathcal{M}$ .

**Definition 4.** Let  $\mathcal{N}$  be a von Neumann subalgebra of  $\mathcal{M}$ . Then there exists a unique trace-preserving faithful normal linear map  $\mathbb{E}_{\mathcal{N}} : \mathcal{M} \rightarrow \mathcal{N}$ .  $\mathbb{E}_{\mathcal{N}}$  satisfies the properties

- (1)  $\mathbb{E}_{\mathcal{N}}$  is completely positive and unital
- (2) For any  $T_1, T_2 \in \mathcal{N}$  and any  $S \in \mathcal{M}$ ,  $\mathbb{E}_{\mathcal{N}}(T_1 S T_2) = T_1 \mathbb{E}_{\mathcal{N}}(S) T_2$ .

The map  $\mathbb{E}_{\mathcal{N}}$  is called the **conditional expectation** of  $\mathcal{M}$  onto  $\mathcal{N}$ .

**Definition 5.** In [2], Brown constructed and proved unique a probability measure  $\nu_T$  supported on a compact subset of  $\text{spec}(T)$  such that for any  $\lambda \in \mathbf{C}$ ,

$$\tau(\log(|T - \lambda|)) = \int_{\mathbf{C}} \log(|z - \lambda|) d\nu_T(z).$$

$\nu_T$  is called the **Brown measure** of  $T$ .

In the case that  $T$  is normal, Brown's construction gives  $\nu_T = \tau \circ E$ , where  $E$  is the projection valued spectral decomposition measure of  $T$ .

The following theorem of Haagerup and Schultz is the cornerstone of our proof.

**Theorem 6.** *Let  $\mathcal{M}$  be a  $II_1$ -factor with trace  $\tau$  and let  $T \in \mathcal{M}$ . For every Borel set  $B \subset \mathbf{C}$ , there exists a unique projection  $P_T(B) \in \mathcal{M}$  such that*

- (1)  $\tau(P_T(B)) = \nu_T(B)$ , where  $\nu_T$  is the Brown measure of  $T$ ,

- (2)  $TP_T(B) = P_T(B)TP_T(B)$ ,
- (3) if  $P_T(B) \neq 0$ , then the Brown measure of  $TP_T(B)$  considered as an element of  $P_T(B)\mathcal{M}P_T(B)$  is concentrated in  $B$  and
- (4) if  $P_T(B) \neq 1$ , then the Brown measure of  $(1 - P_T(B))T$ , considered as an element of  $(1 - P_T(B))\mathcal{M}(1 - P_T(B))$ , is concentrated in  $\mathbf{C} \setminus B$ .

Moreover,  $P_T(B)$  is  $T$ -hyperinvariant and if  $B_1 \subset B_2 \subset \mathbf{C}$  are Borel sets, then  $P_T(B_1) \leq P_T(B_2)$ .

The projection  $P_T(B)$  in Theorem 6 is called the Haagerup-Schultz projection of  $T$  associated to the set  $B$ .

The following two results, from [4] and [5], respectively, will be crucial to the proof of part 3 of theorem 3.

**Lemma 7.** *For any increasing, right-continuous family of  $T$ -invariant projections  $(q_t)_{0 \leq t \leq 1}$  with  $q_0 = 0$  and  $q_1 = 1$ , letting  $\mathcal{D}$  be the von Neumann algebra generated by the set of all the  $q_t$  and  $\mathcal{D}'$  be the relative commutant of  $\mathcal{D}$  in  $\mathcal{M}$ , and letting  $\text{Exp}_{\mathcal{D}'}$  be the  $\tau$  preserving conditional expectation, the Fuglede-Kadison determinants of  $T$  and  $\text{Exp}_{\mathcal{D}'}(T)$  agree. Since the same is true for  $T - \lambda$  and  $\text{Exp}_{\mathcal{D}'}(T) - \lambda$  for all complex numbers  $\lambda$ , we have that the Brown measures of  $T$  and  $\text{Exp}_{\mathcal{D}'}(T)$  agree.*

**Theorem 8.** *If  $T \in \mathcal{M}$ , and if  $p \in \mathcal{M}$  is a projection such that  $Tp = pTp$ , so that we may write  $T = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$ , where  $A = Tp$  and  $C = (1 - p)T$ , then*

$$\Delta_{\mathcal{M}}(T) = \Delta_{p\mathcal{M}p}(A)^{\tau(p)} \Delta_{(1-p)\mathcal{M}(1-p)}(C)^{\tau(1-p)}$$

and

$$\nu_T = \tau(p)\nu_A + \tau(1-p)\nu_C,$$

where  $A$  is considered as an element of  $p\mathcal{M}p$  and  $C$  is considered as an element of  $(1-p)\mathcal{M}(1-p)$ .

**Definition 9.** It was shown in [6] that for any  $T \in \mathcal{M}$ ,  $((T^*)^n T^n)^{1/2n}$  converges in the strong operator topology as  $n$  approaches  $\infty$ . An operator  $T$  is called **s.o.t.-quasinilpotent** if  $((T^*)^n T^n)^{1/2n} \rightarrow 0$  in the strong operator topology as  $n \rightarrow \infty$ .

It was also shown in [6] that  $T$  is s.o.t.-quasinilpotent if and only if the Brown measure of  $T$  is concentrated at 0.

We will also need a characterization from [6] of the Haagerup-Schultz projection of  $T$  associated with the ball  $\overline{B_r} = \{|z| \leq r\}$ .

**10.** Suppose  $\mathcal{M} \leq \mathcal{B}(\mathcal{H})$ . Define a subspace  $\mathcal{H}_r$  of  $\mathcal{H}$  by

$$\mathcal{H}_r = \{\xi \in \mathcal{H} : \exists \xi_n \rightarrow \xi, \text{ with } \limsup_{n \rightarrow \infty} \|T^n \xi_n\|^{1/n} \leq r\}.$$

Then the projection onto  $\mathcal{H}_r$  is equal to  $P_T(\overline{B_r})$ .

### 3. CONSTRUCTION OF THE SPECTRAL MEASURE $E$

Throughout this section,  $\mathcal{M}$ ,  $T$ ,  $\nu_T$ ,  $P_T$  and  $\psi$  will be as described in Theorem 3,  $Z$  will denote  $\{z \in \overline{B_{\|T\|}} : \psi^{-1}(z) \text{ has a minimum}\}$  and  $Y$  will denote  $\overline{B_{\|T\|}} \setminus Z$ .

We first define a Borel measure on the unit interval which will be useful in later proofs.

**Lemma 11.** *Let  $X = \{\min(\psi^{-1}(z)) : z \in \overline{B_{\|T\|}}\}$ . If  $b \subset [0, 1]$  is Borel, then  $\psi(b \cap X)$  is Borel.*

*Proof.* Note first that, for  $t \in (0, 1]$ , we have  $\psi([0, t] \cap X) = \psi([0, t]) \setminus Y$  and  $\psi([0, t) \cap X) = \psi([0, t)) \setminus Y$ , and these sets are Borel. Now, since  $\psi$  restricted to  $X$  is an injection, we have  $\psi((\alpha, \beta) \cap X) = \psi([0, \beta) \cap X) \setminus \psi([0, \alpha) \cap X)$  which is Borel. Since  $[0, 1]$  is second countable, an arbitrary open set  $v = \bigcup_{n \in \mathbf{N}} u_n$  is the countable union of open intervals so that  $\psi(v \cap X) = \psi(\bigcup_{n \in \mathbf{N}} (u_n \cap X)) = \bigcup_{n \in \mathbf{N}} (\psi(u_n \cap X))$  is Borel.

To complete the proof, we show that the collection of sets

$$S = \{b \subset [0, 1] : \psi(b \cap X) \text{ is Borel}\}$$

forms a  $\sigma$ -algebra. Suppose that  $\psi(b \cap X)$  is Borel. Then  $\psi(b^c \cap X) = \psi(X \setminus (b \cap X)) = Z \setminus \psi(b \cap X)$  is Borel. Now suppose that  $(b_n)_{n \in \mathbf{N}} \subset S$ . Then  $\bigcup_{n \in \mathbf{N}} b_n \in S$  by the same argument used for open sets, and we are done.  $\square$

We now define  $\mu(b) = \nu_T(\psi(b \cap X))$  for any Borel set  $b \subset [0, 1]$ . It is clear that  $\mu$  is countably additive, and hence a Borel probability measure on  $[0, 1]$ . That  $\mu$  is a regular measure follows from Theorem 1.1 of [1].

**Observation 12.** For any Borel set  $B \subset \overline{B_{\|T\|}}$ ,  $\mu(\psi^{-1}(B)) = \nu_T(B)$ .

*Proof.* Since  $\psi$  is a bijection from  $X$  to  $Z$  we have

$$\mu(\psi^{-1}(B)) = \nu_T(\psi(\psi^{-1}(B) \cap X)) = \nu_T(B \cap Z) = \nu_T(B)$$

$\square$

Prior to constructing the spectral measure, we will need a map from the open subsets of the closed unit interval to the set of projections in  $\mathcal{M}$ . For an open interval, define

$$\begin{aligned} F(\emptyset) &= 0 \\ F((\alpha, \beta)) &= P_T(\psi([0, \beta))) - P_T(\psi([0, \alpha])) \\ F([0, \beta)) &= P_T(\psi([0, \beta))) \\ F((\alpha, 1]) &= 1 - P_T(\psi([0, \alpha])). \end{aligned}$$

Since  $P_T(\psi([0, t]))$  and  $P_T(\psi([0, t)))$  are increasing in  $t$ , it follows that  $F(u)$  is increasing in  $u$ , and  $F(u_1)F(u_2) = 0$  if  $u_1 \cap u_2 = \emptyset$ . For  $u_1 = (\alpha_1, \beta_1)$  and  $u_2 = (\alpha_2, \beta_2)$  with  $\alpha_1 \leq \alpha_2 \leq \beta_1 \leq \beta_2$ ,

$$\begin{aligned} F(u_1)F(u_2) &= (P_T(\psi([0, \beta_1))) - P_T(\psi([0, \alpha_1]))) (P_T(\psi([0, \beta_2))) - P_T(\psi([0, \alpha_2]))) \\ &= P_T(\psi([0, \beta_1))) - P_T(\psi([0, \alpha_2])) - P_T(\psi([0, \alpha_1])) + P_T(\psi([0, \alpha_1])) \\ &= F(u_1 \cap u_2). \end{aligned}$$

Hence for any open intervals  $u_1$  and  $u_2$ ,  $F(u_1)F(u_2) = F(u_1 \cap u_2)$ .

For an arbitrary open set  $v \subset [0, 1]$ , we first write  $v = \bigcup_{n \in \mathbf{N}} u_n$ , where the  $u_n$  are pairwise disjoint, and all nonempty  $u_n$  are open intervals. Then  $\sum_{n \in \mathbf{N}} F(u_n)$  converges to a projection in the strong operator topology. We define  $F(v) = \sum_{n \in \mathbf{N}} F(u_n)$ . Multiplication of the series and application of the corresponding result for intervals gives us  $F(v_1)F(v_2) = F(v_1 \cap v_2)$  for open sets  $v_1, v_2 \subset [0, 1]$ .

**Observation 13.** For any open set  $v \subset [0, 1]$ ,  $\tau(F(v)) = \mu(v)$ .

*Proof.* For an open interval  $u = (\alpha, \beta)$ , we have

$$\begin{aligned} \tau(F(u)) &= \tau(P_T(\psi([0, \beta])) - P_T(\psi([0, \alpha]))) \\ &= \nu_T(\psi([0, \beta])) - \nu_T(\psi([0, \alpha])) \\ &= \mu([0, \beta]) - \mu([0, \alpha]) \\ &= \mu(u). \end{aligned}$$

The observation follows from additivity of  $\mu$ ,  $F$  and  $\tau$ . □

We are now ready to define the spectral measure  $E$ . For any Borel set  $B \subset \overline{B_{\|T\|}}$ , define

$$E(B) = \bigwedge \{F(v) : v \text{ is open and } \psi^{-1}(B) \subset v\}.$$

Note that  $E$  is increasing and that the range of  $E$  is contained in the von Neumann algebra generated by the projections  $P_T(\psi([0, t]))$  for  $t \in [0, 1]$ , which is commutative. We will prove later that  $E$  defines a spectral measure.

**Proposition 14.** For any Borel set  $B \subset \overline{B_{\|T\|}}$ ,  $\tau(E(B)) = \nu_T(B)$ .

*Proof.* Let  $\epsilon > 0$  be given. There exist open sets  $v_1, v_2 \subset [0, 1]$  such that

- (1)  $\psi^{-1}(B) \subset v_1$  and  $\mu(v_1) - \mu(\psi^{-1}(B)) < \epsilon$ , and
- (2)  $\psi^{-1}(B) \subset v_2$  and  $\tau(F(v_2)) - \tau(E(B)) < \epsilon$ .

Applying Observations 12 and 13 to (1), we have

$$\tau(E(B)) - \nu_T(B) \leq \tau(F(v_1)) - \nu_T(B) = \mu(v_1) - \mu(\psi^{-1}(B)) < \epsilon.$$

Applying Observations 12 and 13 to (2) gives

$$\begin{aligned} \nu_T(B) - \tau(E(B)) &= \mu(\psi^{-1}(B)) - \tau(E(B)) \\ &\leq \mu(v_2) - \tau(E(B)) = \tau(F(v_2)) - \tau(E(B)) < \epsilon. \end{aligned}$$

Hence we have  $|\tau(E(B)) - \nu_T(B)| < \epsilon$ , and we are done. □

**Lemma 15.** If  $B_1$  and  $B_2$  are Borel subsets of  $\overline{B_{\|T\|}}$ , then  $E(B_1)E(B_2) = E(B_1 \cap B_2)$ .

*Proof.* Noting that whenever  $v_1$  is an open set containing  $\psi^{-1}(B_1)$  and  $v_2$  is an open set containing  $\psi^{-1}(B_2)$ ,  $v_1 \cap v_2$  is an open set containing  $\psi^{-1}(B_1) \cap \psi^{-1}(B_2)$ , we have

$$\begin{aligned}
E(B_1 \cap B_2) &= \wedge \{F(v) : v \text{ open}, \psi^{-1}(B_1 \cap B_2) \subset v\} \\
&= \wedge \{F(v) : v \text{ open}, \psi^{-1}(B_1) \cap \psi^{-1}(B_2) \subset v\} \\
&\leq \wedge \{F(v_1 \cap v_2) : v_1, v_2 \text{ open}, \psi^{-1}(B_1) \subset v_1, \psi^{-1}(B_2) \subset v_2\} \\
&= \wedge \{F(v_1)F(v_2) : v_1, v_2 \text{ open}, \psi^{-1}(B_1) \subset v_1, \psi^{-1}(B_2) \subset v_2\} \\
&= \wedge \{F(v_1) : v_1 \text{ open}, \psi^{-1}(B_1) \subset v_1\} \wedge \{F(v_2) : v_2 \text{ open}, \psi^{-1}(B_2) \subset v_2\} \\
&= E(B_1)E(B_2).
\end{aligned}$$

Now let  $\epsilon > 0$  be given. There exist open subsets  $v, \tilde{v}_1, \tilde{v}_2$  of  $[0, 1]$  such that

- (1)  $\psi^{-1}(B_1 \cap B_2) \subset v$  and  $\mu(v \setminus \psi^{-1}(B_1 \cap B_2)) < \epsilon$ ,
- (2)  $a_1 = \psi^{-1}(B_1) \setminus \psi^{-1}(B_1 \cap B_2) \subset \tilde{v}_1$  and  $\mu(\tilde{v}_1 \setminus a_1) < \epsilon$ , and
- (3)  $a_2 = \psi^{-1}(B_2) \setminus \psi^{-1}(B_1 \cap B_2) \subset \tilde{v}_2$  and  $\mu(\tilde{v}_2 \setminus a_2) < \epsilon$ .

Let  $v_i = \tilde{v}_i \cup v$  for  $i = 1, 2$ . Then  $v_1$  is an open set containing  $\psi^{-1}(B_1)$  and  $v_2$  is an open set containing  $\psi^{-1}(B_2)$ . We have

$$\mu(v_1 \cap v_2 \setminus \psi^{-1}(B_1 \cap B_2)) \leq \mu(v \setminus \psi^{-1}(B_1 \cap B_2)) + \mu(\tilde{v}_1 \cap \tilde{v}_2 \setminus \psi^{-1}(B_1 \cap B_2)).$$

Observing that  $a_1 \cap a_2 = \emptyset$  and

$$\tilde{v}_1 \cap \tilde{v}_2 = (a_1 \cap a_2) \cup ((\tilde{v}_1 \setminus a_1) \cap a_2) \cup ((\tilde{v}_2 \setminus a_2) \cap a_1) \cup ((\tilde{v}_1 \setminus a_1) \cap (\tilde{v}_2 \setminus a_2))$$

we have

$$\mu((v_1 \cap v_2) \setminus \psi^{-1}(B_1 \cap B_2)) < 4\epsilon.$$

Applying Observations 12 and 13 and Proposition 14, we have

$$\begin{aligned}
\tau(E(B_1)E(B_2)) - \tau(E(B_1 \cap B_2)) &\leq \tau(F(v_1)F(v_2)) - \tau(E(B_1 \cap B_2)) \\
&= \tau(F(v_1 \cap v_2)) - \tau(E(B_1 \cap B_2)) \\
&< 4\epsilon,
\end{aligned}$$

and we conclude  $E(B_1)E(B_2) = E(B_1 \cap B_2)$ .  $\square$

**Lemma 16.**  *$E$  is countably additive on disjoint sets, where convergence of the series is in the strong operator topology.*

*Proof.* Suppose  $(B_n)_{n \in \mathbf{N}}$  is a countable collection of disjoint Borel subsets of  $\overline{B_{\|T\|}}$ . By claim 7,  $E(B_i)E(B_j) = 0$  if  $i \neq j$ . Then  $E(\bigcup_{n \in \mathbf{N}} B_n)$  is a superprojection of each  $E(B_n)$ , and hence a superprojection of  $\sum_{n \in \mathbf{N}} E(B_n)$ . Also,  $\tau(E(\bigcup_{n \in \mathbf{N}} B_n)) = \nu_T(\bigcup_{n \in \mathbf{N}} B_n) = \tau(\sum_{n \in \mathbf{N}} E(B_n))$ . We conclude  $E(\bigcup_{n \in \mathbf{N}} B_n) = \sum_{n \in \mathbf{N}} E(B_n)$ .  $\square$

We are now ready to show that  $E$  is a spectral measure supported on  $\text{supp}(\nu_T)$ .

*Proof.* We must show three things:

- (1)  $E(\emptyset) = 0$  and  $E(\text{supp}(\nu_T)) = 1$
  - (2)  $E(B_1 \cap B_2) = E(B_1)E(B_2)$  for Borel sets  $B_1, B_2$ , and
  - (3) if  $\mathcal{M}$  acts on a Hilbert space  $\mathcal{H}$ , and  $x, y \in \mathcal{H}$ , then  $\eta(B) = \langle E(B)x, y \rangle$  defines a regular Borel measure on  $\mathbf{C}$ .
- (1) Follows from Proposition 14, since  $\tau(E(\emptyset)) = 0$  and  $\tau(E(\text{supp}(\nu_T))) = 1$ .

- (2) Was proven as Lemma 15.  
 (3) That  $\eta$  is countably additive on disjoint sets follows from Lemma 15. Regularity of  $\eta$  follows from Theorem 1.1 of [1].

□

#### 4. PROOF OF THEOREM 3

We first establish several results which will be used to prove Part 3. Throughout this section,  $\mathcal{M}$ ,  $T$ , and  $\psi$  are as described in Theorem 3, and  $\mu$ ,  $E$  and  $E_v$  are as defined in Section 3.  $\mathcal{M}$  acts on a Hilbert space  $H$ .

We now show that  $\int_{\mathbf{C}} z dE$  is the norm limit of conditional expectations onto an increasing sequence of abelian von Neumann algebras. For each  $n$ , divide the  $3\|T\|$  by  $3\|T\|$  square centered at 0 into  $2^n$  by  $2^n$  squares of equal size indexed  $(A_{n,k})_{k=1}^{2^{2n}}$ ,  $k$  increasing to the right then down. Include in each  $A_{n,k}$  the top and left edge, excluding the bottom-left and top-right corners, so that for each  $n$ ,  $A_{n,k} \cap A_{n,j} = \emptyset$  whenever  $j \neq k$  and  $\overline{B_{\|T\|}} \subset \cup_{k=1}^{2^{2n}} A_{n,k}$ . Let  $D_n$  be the von Neumann algebra generated by the (orthogonal) projections  $(E(A_{n,k}))_{k=1}^{2^{2n}}$ .

**Proposition 17.** *Let  $\mathbb{E}_{D_n}(T)$  denote the conditional expectation of  $T$  onto  $D_n$ . Then  $\mathbb{E}_{D_n}(T)$  converges in norm as  $n \rightarrow \infty$  to  $\int_{\mathbf{C}} z dE$ .*

*Proof.* Observe that

$$\mathbb{E}_{D_n}(T) = \sum_{\substack{1 \leq k \leq 2^{2n} \\ \tau(E(A_{n,k})) \neq 0}} \frac{\tau(E(A_{n,k})TE(A_{n,k}))}{\tau(E(A_{n,k}))} E(A_{n,k}).$$

Applying Brown's analog of Lidskii's theorem (see [2]) gives

$$\mathbb{E}_{D_n}(T) = \sum_{\substack{1 \leq k \leq 2^{2n} \\ \nu_T(A_{n,k}) \neq 0}} \frac{\int_{A_{n,k}} z d\nu_T(z)}{\nu_T(A_{n,k})} E(A_{n,k}).$$

For each  $n$ , define

$$f_n(w) = \sum_{\substack{1 \leq k \leq 2^{2n} \\ \nu_T(A_{n,k}) \neq 0}} \frac{\int_{A_{n,k}} z d\nu_T(z)}{\nu_T(A_{n,k})} \chi_{A_{n,k}}(w) + \sum_{\substack{1 \leq k \leq 2^{2n} \\ \nu_T(A_{n,k}) = 0}} \frac{\int_{A_{n,k}} z dm(z)}{m(A_{n,k})} \chi_{A_{n,k}}(w),$$

where  $m$  is the Lebesgue measure on  $\mathbf{C}$ .

Since  $\nu_T(A_{n,k}) = 0$  implies  $E(A_{n,k}) = 0$ ,  $\int_{\mathbf{C}} f_n dE = \mathbb{E}_{D_n}(T)$ . Note that  $f_n$  converges uniformly on  $\text{supp}(E)$  to the inclusion function  $f(z) = z$ . Hence  $\int_{\mathbf{C}} f_n dE$  converges in norm to  $\int_{\mathbf{C}} z dE$ , and we are done. □

Let  $D$  be the von Neumann algebra generated by  $(E(\psi([0, t])))_{t \in [0, 1]}$  (or equivalently by  $\bigcup_{n=1}^{\infty} D_n$ ).

**Proposition 18.** *Suppose that  $T \in D'$  and  $B \subset \overline{B_{\|T\|}}$  is Borel with  $\nu_T(B) \neq 0$ . Then the Brown measure of  $E(B)TE(B)$ , considered as an element of  $E(B)\mathcal{M}E(B)$ , is concentrated in  $B$ .*

*Proof.* We begin by observing that for any open  $v \subset [0, 1]$ , with  $\tau(F(v)) \neq 0$ ,  $F(v) \in D$  and if  $v = (\alpha, \beta)$  is an open interval, then  $\nu_{TF(v)}$  is concentrated in  $\psi([0, \beta)) \setminus \psi([0, \alpha])$ , and hence is also concentrated in  $\psi((\alpha, \beta)) \cap Z$ , where  $Z$  is as described in Section 3. Thus  $\nu_{TF(v)}$  is concentrated in  $\psi((\alpha, \beta) \cap X)$ .

Now suppose that  $v = \bigcup_{n=1}^{\infty} u_n$  where all nonempty  $u_n$  are pairwise disjoint open intervals. Let  $\epsilon > 0$  be given. Let  $N$  be so large that

$$\tau \left( \sum_{n=1}^N F(u_n) \right) > \tau(F(v))(1 - \epsilon).$$

Then, since each  $F(u_n)$  commutes with  $T$ , Theorem 8 gives

$$\nu_{TF(v)} = \frac{1}{\tau(F(v))} \left( \sum_{n=1}^N \tau(F(u_n)) \nu_{TF(u_n)} + \tau \left( \sum_{n=N+1}^{\infty} F(u_n) \right) \nu_{(\sum_{n=N+1}^{\infty} F(u_n))T} \right).$$

Hence, since each  $\nu_{TF(u_n)}$  is concentrated in  $\psi(u_n \cap X) \subset \psi(v \cap X)$ , we have

$$\nu_{TF(v)}(\psi(v \cap X)) \geq \frac{1}{\tau(F(v))} \left( \sum_{n=1}^N \tau(F(u_n)) \right) \nu_{TF(u_n)}(\psi(v \cap X)) > 1 - \epsilon,$$

so that  $\nu_{TF(v)}$  is concentrated in  $\psi(v \cap X)$ .

Now observe that when  $v$  is an open set containing  $\psi^{-1}(B)$ , since

$$\nu_{TF(v)} = \frac{1}{\tau(F(v))} (\tau(E(B)) \nu_{TE(B)} + \tau(F(v) - E(B)) \nu_{(F(v) - E(B))T}),$$

$\nu_{TE(B)}$  is concentrated in  $\psi(v \cap X)$ .

Choose an open set  $v \subset [0, 1]$  such that  $\psi^{-1}(B) \subset v$  and  $\mu(v) - \mu(\psi^{-1}(B)) < \epsilon$ . Then using Theorem 7 and Observation 11,

$$\begin{aligned} \epsilon &> \nu_T(\psi(v \cap X)) - \nu_T(B) \\ &= \tau(E(B)) \nu_{TE(B)}(\psi(v \cap X) \setminus B) + (1 - \tau(E(B))) \nu_{(1-E(B))T}(\psi(v \cap X) \setminus B) \\ &\geq \tau(E(B)) \nu_{TE(B)}(\psi(v \cap X) \setminus B). \end{aligned}$$

Hence

$$\tau(E(B)) - \epsilon < \tau(E(B))(1 - \nu_{TE(B)}(\psi(v \cap X) \setminus B)) = \tau(E(B))(\nu_{TE(B)}(B)).$$

Thus

$$1 - \frac{\epsilon}{\tau(E(B))} < \nu_{TE(B)}(B).$$

Letting  $\epsilon$  tend to 0 gives the desired result.  $\square$

**Lemma 19.** *If  $T \in D'$ , then the Brown measure of  $T - \mathbb{E}_{D_n}(T)$  is supported in the ball of radius  $\frac{6\sqrt{2}\|T\|}{2^n}$ .*

*Proof.* The key observation is that for any  $\alpha \in \mathbf{C}$ , if  $\nu_{T-\alpha}$  is the Brown measure of  $T-\alpha$ , then for any Borel set  $B \subset \mathbf{C}$ ,  $\nu_{T-\alpha}(B) = \nu_T(B-\alpha)$ . Since whenever  $E(A_{n,k}) \neq 0$  the Brown measure of  $TE(A_{n,k})$  is supported in  $A_{n,k}$ , the Brown measure of  $(T - \frac{\tau(TE(A_{n,k}))}{\tau(E(A_{n,k}))})E(A_{n,k})$  is supported in the square centered at 0 with edge length  $\frac{6\|T\|}{2^n}$ . We



complete the proof by observing that  $T - \mathbb{E}_{D_n}(T) = \sum_{k=1}^{2^{2n}} \left( T - \frac{\tau(TE(A_{n,k}))}{\tau(E(A_{n,k}))} \right) E(A_{n,k})$  and applying Theorem 8 to compute the Brown measure of the sum.  $\square$

We now are ready to prove Theorem 3.

*Proof.* (1) Whenever  $v$  is an open set containing  $\psi^{-1}(\psi([0, t]))$ , there exists  $\epsilon > 0$  such that  $[0, t + \epsilon) \subset v$  so we see that

$$P_T(\psi([0, t])) \leq F([0, t + \epsilon)) \leq F(v).$$

Hence we see that

$$P_T(\psi([0, t])) \leq E(\psi([0, t])).$$

By Proposition 14 and Theorem 6,

$$\tau(P_T(\psi([0, t]))) = \tau(E(\psi([0, t])))$$

so that

$$P_T(\psi([0, t])) = E(\psi([0, t])).$$

(2) Was proven as Proposition 14.

(3) We show this first in the case that  $T \in D'$ . Observe from the proof of Proposition 17 that  $\|\mathbb{E}_D(T) - \mathbb{E}_{D_n}(T)\| \leq \frac{3\sqrt{2}\|T\|}{2^n}$ . The rest of this argument is taken from the proof of Lemma 24 in [3].

We assume without loss of generality that  $\|T\| \leq 1/2$ . Fix  $n \in \mathbb{N}$  and a unit vector  $\xi \in H$ . By assumption  $T \in D'$ , so we have

$$(T - \mathbb{E}_D(T))^{2m} = \sum_{k=0}^{2m} (-1)^k \binom{2m}{k} (\mathbb{E}_D(T) - \mathbb{E}_{D_n}(T))^{2m-k} (T - \mathbb{E}_{D_n}(T))^k.$$

Since  $\|T\| \leq 1/2$ , both  $\mathbb{E}_D(T) - \mathbb{E}_{D_n}(T)$  and  $T - \mathbb{E}_{D_n}(T)$  are contractions. For  $k \leq m$  and any  $\eta \in H$ , we have

$$\|(\mathbb{E}_D(T) - \mathbb{E}_{D_n}(T))^{2m-k} (T - \mathbb{E}_{D_n}(T))^k \eta\|_H \leq \|\mathbb{E}_D(T) - \mathbb{E}_{D_n}(T)\|^m.$$

For  $k > m$  and any  $\eta \in H$  we have

$$\|(\mathbb{E}_D(T) - \mathbb{E}_{D_n}(T))^{2m-k} (T - \mathbb{E}_{D_n}(T))^k \eta\|_H \leq \|(T - \mathbb{E}_{D_n}(T))^m \eta\|_H.$$

Hence for any  $\eta \in H$ ,

$$\|(T - \mathbb{E}_D(T))^{2m} \eta\|_H \leq 2^{2m} \max \left\{ \left( \frac{3\sqrt{2}\|T\|}{2^n} \right)^m, \|(T - \mathbb{E}_{D_n}(T))^m \eta\|_H \right\}. \quad (1)$$

By Lemma 19, the Brown measure of  $T - \mathbb{E}_{D_n}(T)$  is supported in the ball of radius  $\frac{6\sqrt{2}\|T\|}{2^n}$  centered at 0. By the Haagerup-Schultz characterization (10), there exists a sequence  $\xi_m \rightarrow \xi$  such that  $\|\xi_m\|_H = 1$  and

$$\limsup_{m \rightarrow \infty} \|(T - \mathbb{E}_{D_n}(T))^m \xi_m\|_H^{1/m} \leq \frac{6\sqrt{2}\|T\|}{2^n}.$$

Hence there exists  $M$  (depending on  $n$ ) such that

$$\|(T - \mathbb{E}_{D_n}(T))^m \xi_m\|_H \leq \left( \frac{7\sqrt{2}\|T\|}{2^n} \right)^m, \quad m > M.$$

Taking  $\eta = \xi_m$  in (1), we have

$$\|(T - \mathbb{E}_D(T))^{2m} \xi_m\|_H^{1/m} \leq \frac{28\sqrt{2}\|T\|}{2^n}, \quad m > M.$$

Since  $\xi$  was arbitrary, it follows from characterization (10) that the Brown measure of  $(T - \mathbb{E}_D(T))^2$  is supported in the ball of radius  $\frac{28\sqrt{2}\|T\|}{2^n}$  centered at 0. Letting  $n \rightarrow \infty$ , we obtain that the Brown measure of  $T - \mathbb{E}_D(T)$  is  $\delta_0$ .

For  $T \notin D'$ , we first show that  $P_T(\psi([0, t])) = P_{\mathbb{E}_{D'}(T)}(\psi([0, t]))$  for all  $t \in [0, 1]$ . For any  $t$ ,  $P_T(\psi([0, t])) \in D$ , so

$$\mathbb{E}_{D'}(T)P_T(\psi([0, t])) = P_T(\psi([0, t]))\mathbb{E}_{D'}(T)P_T(\psi([0, t])).$$

By Lemma 7,  $T$  and  $\mathbb{E}_{D'}(T)$  have the same Brown measure, so we have for all  $t$

$$\tau(P_T(\psi([0, t]))) = \nu_T(\psi([0, t])) = \nu_{\mathbb{E}_{D'}(T)}(\psi([0, t])).$$

For any  $s, t \in [0, 1]$   $P_T(\psi([0, s]))$  is  $TP_T(\psi([0, t]))$  invariant, so by Lemma 7  $TP_T(\psi([0, t]))$  and  $\mathbb{E}_{D'}(TP_T(\psi([0, t])))$  have the same Brown measure for any  $t$ , so whenever  $P_T(\psi([0, t])) \neq 0$  we have

$$\nu_{\mathbb{E}_{D'}(T)P_T(\psi([0, t]))} = \nu_{\mathbb{E}_{D'}(TP_T(\psi([0, t])))} = \nu_{TP_T(\psi([0, t]))}$$

is supported in  $\psi([0, t])$ . Similarly  $P_T(\psi([0, s]))$  is  $(1 - P_T(\psi([0, t])))T$  invariant for all  $s, t \in [0, 1]$ , so  $(1 - P_T(\psi([0, t])))T$  and  $\mathbb{E}_{D'}((1 - P_T(\psi([0, t])))T) = (1 - P_T(\psi([0, t])))\mathbb{E}_{D'}(T)$  have the same Brown measure, which is supported in  $\mathbf{C} \setminus \psi([0, t])$  whenever  $P_T(\psi([0, t])) \neq 1$ . Hence by Theorem 6  $P_T(\psi([0, t]))$  is the Haagerup-Schultz projection of  $\mathbb{E}_{D'}(T)$  associated with the set  $\psi([0, t])$ .

Since  $P_T(\psi([0, t])) = P_{\mathbb{E}_{D'}(T)}(\psi([0, t]))$  for all  $t \in [0, 1]$ , we see that  $\psi$  generates the same spectral measure  $E$  and abelian subalgebra  $D$  for both  $T$  and  $\mathbb{E}_{D'}(T)$ . Applying Lemma 7 we have  $T - \int_{\mathbf{C}} z dE$  and  $\mathbb{E}_{D'}(T) - \int_{\mathbf{C}} z dE$  have the same Brown measure, which we have shown is  $\delta_0$ . □

## REFERENCES

- [1] P. Billingsley, *Convergence of Probability Measures*, John Wiley and Sons, New York, 1968.
- [2] L. G. Brown, *Lidskii's theorem in the type II case*, Geometric methods in operator algebras (Kyoto, 1983), Pitman Res. Notes Math. Ser., vol. 123, Longman Sci. Tech., Harlow, 1986, pp. 1–35.
- [3] K. Dykema, F. Sukochev, and D. Zanin, *A decomposition theorem in  $II_1$ -factors*, J. reine angew. Math., to appear, available at <http://arxiv.org/abs/1302.1114>.
- [4] ———, *Holomorphic Functional Calculus on Upper Triangular Forms in Finite von Neumann Algebras*, preprint, available at <http://arxiv.org/abs/1310.2524>.
- [5] U. Haagerup and H. Schultz, *Brown measures of unbounded operators affiliated with a finite von Neumann algebra*, Math. Scand. **100** (2007), 209–263.
- [6] ———, *Invariant subspaces for operators in a general  $II_1$ -factor*, Publ. Math. Inst. Hautes Études Sci. **109** (2009), 19–111.

- [7] F. Zheng, *Matrix Theory: Basic results and techniques*, Second edition, Universitext, Springer, New York, 2011.

DEPARTMENT OF MATHEMATICS, TEXAS A&M UNIVERSITY, COLLEGE STATION, TX, USA.  
*E-mail address:* `jnoles@math.tamu.edu`